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A UNIQUENESS THEOREM FOR REPRESENTATIONS OF GSO(6) AND THE STRONG MULTIPLICITY ONE THEOREM FOR GENERIC REPRESENTATIONS OF GSp(4)

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ABSTRACT
Consider the θ-correspondence from GSp(4) to GSO(6). We prove that locally over a nonarchimedean field F, this correspondence is injective on generic representations (i.e. with Whittaker model) of GSp(4, F). We use this to show the strong multiplicity one property for irreducible, automorphic, cuspidal representations of GSp(4, A), which are generic.

Introduction

Let $G = \mathrm{GSO}(6)$, the connected component of the group of similitudes of a split quadratic form in six variables. Let F be a local nonarchimedean field. Our main theorem (Theorem 2.1) says that for an irreducible, admissible representation σ of G_F, the space of certain linear functionals is at most one dimensional. Let us describe this space. Write the elements of G as matrices $g \in \mathrm{GL}(6)$, satisfying ${}^t g w_6 g = \mu(g) w_6$,

$$w_6 = \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \end{bmatrix}.$$

Let L be the space of column vectors in four dimensions, equipped with the quadratic form defined by

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$$w_4 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Let $e \in L$ satisfy ${}^t e w_4 e = 1$. Consider the following subgroup R of G:

$$R = \left\{ r = \begin{bmatrix} 1 & * & * \\ & h & v \\ & & 1 \end{bmatrix} \in G \;\middle|\; \begin{matrix} he = e \\ v \in L \end{matrix} \right\}.$$

Define $\chi_0(r) = {}^t v w_4 e$. This is a rational character of R. Let ψ be a nontrivial character of F. The above space of linear functionals is

$$L_{\chi_0,\psi} = \{ l_\psi \in V_\sigma^* \mid l_\psi(\sigma(r)v) = \psi(\chi_0(r)) l_\psi(v);\ v \in V_\sigma,\ r \in R \}$$

and the assertion is that $L_{\chi_0,\psi}$ is at most one dimensional.

The functionals l_ψ arise in the following situation. Consider the "dual pair" (GSp(4), GSO(6)). Consider the local θ correspondence, and let $\theta(\sigma)$ be the set of equivalence classes of irreducible representations of GSp$(4, F)$ which correspond to σ under the local θ-map, then we show that the number of generic elements of $\theta(\sigma)$ (i.e. those with standard Whittaker model) is less than or equal to dim $L_{\chi_0,\psi}$ (Theorem 1.3). Thus σ has at most one generic θ-lift to GSp$(4, F)$. We remark that Rallis [R] proved the Howe duality conjecture for many cases, and our case is not one of them (and also (GSp(4), GSO(6)) is not exactly a dual pair). We prove the uniqueness theorem in section two. We use the Gelfand–Kazhdan method (explained in [B.Z] part III).

H. Jacquet, I. Piatetski-Shapiro and J. Shalika show in a work in preparation [J. PS. S] that under the global θ-correspondence from GSp(4) to GSO(6), irreducible, automorphic cuspidal representations of GSp$(4, A)$ (A—the adeles of a global field k), which are generic, have a nonzero image (and only these). Also an irreducible, automorphic, cuspidal representation σ of GSO$(6, A)$ is in the image of the θ-correspondence from GSp (4) if

$$\int_{R_k \backslash R_A} \varphi(r) \psi^{-1}(\chi_0(r)) dr \not\equiv 0, \qquad \varphi \in \sigma$$

(ψ is a nontrivial character of $k \backslash A$. In the definition of χ_0, we choose $e \in L_k$). This explains the global set up (Propositions 1.1, 1.2). Now, GSO(6) is up to multiples by elements of the center, the same as $\{\pm I_4\} \backslash$ GL(4), and we can use the properties of the above θ-correspondence to consider the question of the strong multiplicity one theorems for generic representations of GSp$(4, A)$.

Using the similar property GL(n) [J. Sh] and the multiplicity one theorem for GL(n) [Sh] and for GSp(4), for generic representations [PS], one immediately reduces the question to one of the injectivity of the above θ-map. Locally in the archimedean case, the injectivity is proved in [J. PS. S], and, as mentioned above, in the nonarchimedean case it is proved here.

§0. Notations and preliminaries

1. Let F be a field (Char $F \neq 2$). Put

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

then

$$\mathrm{GSp}(2n, F) = \{g \in M(2n, F) \mid {}^t g J g = \mu(g) J, \qquad \mu(g) \in F^*\}.$$

2. Let F be a field and X a finite dimensional vector space over F, equipped with a nondegenerate symmetric form (,), then we denote

$$\mathrm{GO}(X) = \{g \in \mathrm{GL}(X) \mid (gx_1, gx_2) = \mu(g)(x_1, x_2);\ \forall x_1, x_2 \in X, \mu(g) \in F^*\}.$$

We denote the connected component of GO(X) by GSO(X) and the subgroup of those g in GSO(X) with $\mu(g) = 1$, by SO(X). The groups of this type that we encounter here are with dim $X = 6$, 4 with a split form, so we denote them for short GSO(6), GSO(4) respectively.

For any field F we have an injection

$$Z_2 \backslash \mathrm{GL}(4, F) \overset{l}{\hookrightarrow} \mathrm{GSO}(6, F)$$

and if C denotes the center of GSO($6, F$) then GSO($6, F$) $= C \cdot \mathrm{Im}\, l$. The injection l is defined as follows. Let GL($4, F$) act from the left on the four dimensional space V. The space $X = \Lambda^2 V$ is six dimensional. Let $\varepsilon_1, \ldots, \varepsilon_4$ be a basis for V over F. The space $\Lambda^4 V$ is one dimensional. The form on $X \times X$ defined by $v_1 \wedge v_2 \wedge u_1 \wedge u_2 = (v_1 \wedge v_2, u_1 \wedge u_2)\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$ is symmetric, nondegenerate and splits over F. The injection l is defined by $g \to \Lambda^2 g$, $g \in Z_2 \backslash \mathrm{GL}(4, F)$. Note that $\Lambda^2 g$ preserves (,) up to det g.

3. Let F be a local field and ψ a nontrivial character of F. Let Sp($2n, F$) act from the right on Z, a $2n$ dimensional space over F, preserving the symplectic form $\langle\ ,\ \rangle$. Let $Z = Z^+ + Z^-$ be a polarization of Z, that is Z^+, Z^- are maximal isotropic subspaces of Z. Let ω_ψ be the (smooth) Weil representation of $\widetilde{\mathrm{Sp}}(2n, F)$, corresponding to ψ. It acts in $S(Z^+)$, the Schwartz–Bruhat

functions of Z^+. There is also an adelic analogue. For details, see for example, [H. PS], where also the notions of reductive dual pairs and the local and global θ-correspondences are explained. Here we need only the following modified case.

4. We consider the "pair" (GSp(4), GSO(6)). Let GSp(4) act on the four dimension space Y, preserving up to nonzero scalars the symplectic form $\langle\ ,\ \rangle$. Let GSO(6) act on the six dimensional space X preserving up to nonzero scalars the quadratic form (,). The space $Z = Y \otimes X$ is symplectic of dimension 24, with symplectic form $\langle\ ,\ \rangle \otimes (\ ,\ )$, and we have a homomorphism $\mathrm{GSp}(4) \times \mathrm{GSO}(6) \to \mathrm{GSp}(Z)$ with kernel $\{tI_4, t^{-1}I_6) \mid t \neq 0\}$. Let F be a local field. We modify ω_ψ to be a representation ω of $\mathrm{GSp}(4,F) \times \mathrm{GSO}(6,F)$. Let $Z = Z^+ + Z^-$ be a polarization. The space of ω is $S(Z^+ \times F^*)$, the Schwartz–Bruhat functions on $Z^+ \times F^*$. For $\phi \in S(Z^+ \times F^*)$, set $\phi_t(z^+) = \phi(z^+, t)$. For $a = (g, I)$ or $a = (I, h)$ in $\mathrm{GSp}(4) \times \mathrm{GSO}(6)$ with similitude factors 1, we define

$$(\omega(a)\phi)(z^+, t) = (\omega_{\psi^t}(a)\phi_t)(z^+)$$

and for an element a of the form

$$\begin{pmatrix} I_{12} & \\ & yI_{12} \end{pmatrix}$$

(in GSp(24, F)),

$$(\omega(a)\phi)(z^+, t) = \phi(z^+, ty^{-1}).$$

We construct in a similar fashion the representation in the adelic case. We also construct the θ-series and the θ-lifts. Let k be a global field and A its ring of adeles, then for $\phi \in S(Z_A^+ \times A^*)$, we define

$$\theta^\phi(g, h) = \sum_{z^+ \in Z_k^+,\, t \in k^*} \omega(g, h)\phi(z^+, t); \qquad g \in \mathrm{GSp}(4, A), \quad h \in \mathrm{GSO}(6, A)$$

and for a cusp form φ on $\mathrm{GSO}(6, A)$ the function

$$g \to \int_{\mathrm{GSO}(6,k)\backslash \mathrm{GSO}(6,A)} \theta^\phi(g, h)\varphi(h)dh$$

defines an automorphic form on $\mathrm{GSp}(4, A)$. When φ varies in an irreducible, automorphic, cuspidal representation of $\mathrm{GSO}(6, A)$ and ϕ varies in $S(Z_A^+ \times A^*)$, these forms generate an automorphic representation of $\mathrm{GSp}(4, A)$, denoted by $\theta(\sigma)$. (Similarly in the other direction.)

5. Let k be a global field. An automorphic representation π of $GSp(4, A)$ is said to be generic if

$$\int_{U_k \backslash U_A} \varphi(u)\psi^{-1}(u)du \neq 0, \qquad \varphi \in \pi$$

where

$$U = \left\{ u = \begin{bmatrix} 1 & x & a & b \\ & 1 & c & y \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \in GSp(4) \right\}$$

ψ is a nontrivial character of $k\backslash A$ and $\psi(u) = \psi(x + y)$.

Let F be a local field. An admissible representation π of $GSp(4, F)$ is said to be generic if there is a linear functional l on the space of π, V_π, satisfying

$$l(\pi(u)v) = \psi(u)l(v), \qquad v \in V_\pi, \quad u \in U.$$

l is called a Whittaker functional for π. (If F is archimedean l is assumed to be continuous in the C^∞ topology.)

§1. Motivations and applications

We describe how the functional l_ψ comes into play when considering θ-correspondence between GSp(4) and GSO(6). This is one of the subjects of [J. PS. S].

Let F be a field and X a six dimensional vector space regarded as an algebraic group over F. Assume that X is equipped with a nondegenerate symmetric form (,), which splits over F. Write

(1.1) $$X = \mathrm{Span}\{e_0\} + L + \mathrm{Span}\{e_{-0}\}$$

where $e_{\pm 0} \in X_F$ are isotropic, $(e_0, e_{-0}) = 1$ and L is the orthogonal complement of $\mathrm{Span}\{e_0\} + \mathrm{Span}\{e_{-0}\}$. Denote by GSO(6) the connected component of the group of similitudes of $(X, (,))$. We let GSO(6) act on X from the left and we write its elements as matrices according to the decomposition (1.1). Consider the subgroup

(1.2) $$R = \left\{ r = \begin{bmatrix} 1 & {}^t\mathbf{u} & z \\ & h & \mathbf{v} \\ & & 1 \end{bmatrix} \in GSO(6) \,\middle|\, he = e \right\}$$

where $e \in L_F$ is fixed such that $(e, e) = 1$. In (1.2) we identify v with a general element of L, $h \in \mathrm{SO}(L)$ and ${}^t u$ is the transposed of $-h^{-1} \cdot v$, i.e., the element of L^* which sends $l \in L$ to $-(h^{-1}v, l)$. Consider the following rational homomorphism from R to F (notation of (1.2)),

$$\chi_0 : r \rightarrow (v, e). \tag{1.3}$$

Let k be a global field and A its ring of adeles. Let ψ be a nontrivial character of $k \backslash A$. Let σ be an irreducible, automorphic cuspidal representation of $\mathrm{GSO}(6, A)$. Define for $\varphi \in \sigma$

$$l_\psi(\varphi) = \int_{R_k \backslash R_A} \varphi(r) \psi^{-1}(\chi_0(r)) dr.$$

(This integral converges absolutely since φ is a cusp form.) Denote by $\theta(\sigma)$ the automorphic representation of $\mathrm{GSp}(4, A)$ obtained by the θ-correspondence (see section 0). The functional l_ψ arises when we consider the case where $\theta(\sigma)$ is generic.

PROPOSITION 1.1. *Assume $\theta(\sigma)$ is generic, then l_ψ is nontrivial on σ.*

PROOF. Let ψ be a nontrivial character of $k \backslash A$. We know that the following Fourier coefficient is nontrivial on $\theta(\sigma)$,

$$W_\xi = \int_{U_k \backslash U_A} \psi^{-1}(u) \xi(u) du, \qquad \xi \in \theta(\sigma), \tag{1.4}$$

Let $\xi(g) = \int_{\mathrm{GSO}(6,k) \backslash \mathrm{GSO}(6,A)} \theta^\phi(g, h) dh$; $g \in \mathrm{GSp}(4, A)$, $f \in \sigma$. We realize the action of the Weil representation ω on the space $S(Z_A^+ \times A^*)$ where $Z^+ = Y^+ \otimes X = X \oplus X$, and Y^+ is a maximal isotropic subspace of the four dimensional symplectic space Y. The formulas we need are as follows. Let $\phi \in S(Z_A^+ \times A^*)$, $h \in \mathrm{GSO}(6, A)$, $\mu(h)$-the similitude factor of h, $g \in \mathrm{GL}(2, A)$, $y \in A^*$ and $S = {}^t S$. Then

$$\omega(1, h)\phi(x_1, x_2; t) = |\mu(h)|^{-3} \phi(h^{-1}x_1, h^{-1}x_2; \mu(h)t),$$

$$\omega\left(\begin{pmatrix} g & 0 \\ 0 & y\,{}^t g^{-1} \end{pmatrix}, 1\right) \phi(x_1, x_2; t) = |\det g|^3 \phi((x_1, x_2)g; y^{-1}t), \tag{1.5}$$

$$\omega\left(\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}, 1\right) \phi(x_1, x_2; t) = \psi^t(\mathrm{tr}\, \mathrm{Gr}(x_1, x_2)S) \phi(x_1, x_2; t),$$

$\mathrm{Gr}(x_1, x_2)$ is the matrix $((x_i, x_j))$, $1 \leqq i, j \leqq 2$. We first compute the partial Fourier coefficient

$$W'_\xi(g)=\int_{V_k\backslash V_A}\psi^{-1}(v)\xi(vg)dv;\qquad V=\left\{\begin{pmatrix}I_2 & S\\ 0 & I_2\end{pmatrix}\,\middle|\,{}^tS=S\right\}.$$

Using (1.5) (and the definitions in section 0), we get

$$W'_\xi(g)=\int_{\mathrm{GSO}(6,k)\backslash \mathrm{GSO}(6,A)}\sum_{(x_1,x_2;t)\in X_0}\omega(g,h)\phi(x_1,x_2;t)f(h)dh \tag{1.6}$$

where

$$X_0=\left\{(x_1,x_2;t)\in X_k^2\times k^*\,\middle|\,t\,\mathrm{Gr}(x_1,x_2)=\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}.$$

X_0 is the union of two orbits O_0, O_1 under GSO(6, k). The elements of O_0 are of the form $(0, x; t)$, and those of O_1 have the property dim Span$\{x_1, x_2\}=2$.We have

$$W_\xi=\int_{k\backslash A}\psi^{-1}(u)W'_\xi\begin{bmatrix}1&u&&\\0&1&&\\&&1&\\&&-u&1\end{bmatrix}du. \tag{1.7}$$

It is easy to see that the contribution of O_0 to W_ξ is zero. For O_1, pick the representative $(e_0, e; 1)$. Its stabilizer in GSO(6) is

$$R'=\left\{\begin{bmatrix}1&*&*\\&h&v\\&&1\end{bmatrix}\in \mathrm{GSO}(6)\,\middle|\,\begin{matrix}(v,e)=0\\ h\cdot e=e\end{matrix}\right\}.$$

Using this and substituting (1.6) in (1.7) we get

$$W_\xi=\int_{k\backslash A}\psi^{-1}(u)\int_{R'_A\backslash \mathrm{GSO}(6,A)}\omega(1,h)\phi(e_0,ue_0+e;1)\int_{R'_k\backslash R'_A}f(r'h)dr'dhdu.$$

Now note that

$$r_u=\begin{bmatrix}1&u{}^te&-\frac{1}{2}u^2\\&I&-ue\\&&1\end{bmatrix}$$

has the property $r_u^{-1}\cdot(e_0,e;1)=(e_0,ue_0+e;1)$, and so $\omega(1,r_uh)\phi(e_0,e;1)=\omega(1,h)\phi(e_0,ue_0+e;1)$. Changing variables in h, we obtain

$$W_\xi = \int_{R'_A \backslash \mathrm{GSO}(6,A)} \omega(1,h)\phi(e_0, e; 1) \int_{R_k \backslash R_A} \psi^{-1}(\chi_0(r)) f(rh) dr dh$$

$$= \int_{R'_A \backslash \mathrm{GSO}(6,A)} \omega(1,h)\phi(e_0, e; 1) l_\psi(\sigma(h)f) dh.$$

Since $W_\xi \not\equiv 0$, then l_ψ is nontrivial on σ. □

In a similar fashion one proves

PROPOSITION 1.2. *Let π be an irreducible, automorphic, cuspidal representation of* GSp(4, A). *Assume that π is generic. Then the θ-lift of π, $\theta(\pi)$ to* GSO(6, A) *has a Whittaker model (and in particular $\theta(\pi) \neq 0$).*

PROOF (sketch). As in Proposition 1.1, compute the Whittaker Fourier coefficient W_ξ of an element $\xi \in \theta(\pi)$. This time realize ω in $S(Z_A^+ \times A^*)$ and $Z^+ = Y \otimes X^+$, where X^+ is a maximal isotropic subspace of X. Now GSp(4, A) acts linearly. We get for W_ξ a formula similar to the one in the end of the proof of Proposition 1.1. If

$$\xi(h) = \int_{\mathrm{GSp}(4,k) \backslash \mathrm{GSp}(4,A)} \theta^\phi(g,h)\varphi(g) dg, \qquad \varphi \in \pi$$

then

$$W_\xi = \int_{H_A \backslash \mathrm{GSp}(4,A)} \omega(g,1)\phi(z_0; 1) w_\varphi(g) dg;$$

z_0 is a certain point in Z_k^+, H is the stabilizer of z_0, and w_φ is the Whittaker function of φ. Now it is possible to see that this integral does not vanish identically. □

REMARK. Propositions 1.1 and 1.2 are parts of the following more complete theorem.

THEOREM ([J. PS. S.]). (i) *Let π be an irreducible, automorphic, cuspidal representation of* GSp(4, A), *then $\theta(\pi)$, the θ-lift to* GSO(6, A), *is nonzero iff π is generic.*

(ii) *Let σ be an irreducible, automorphic, cuspidal representation of* GSO(6, A), *then $\sigma = \theta(\pi)$, for an irreducible, automorphic, cuspidal representation π of* GSp(4, A) *iff l_ψ is nontrivial on σ.*

We now turn to the local analogue of Proposition 1.1.

Let F be a local nonarchimedean field and σ an irreducible admissible representation of GSO(6, F). Let ψ be a nontrivial character of F. We consider for σ linear functionals l_ψ on the space of σ, V_σ, satisfying

$$l_\psi(\sigma(r)v) = \psi(\chi_0(r))l_\psi(v), \qquad r \in R, \quad v \in V_\sigma. \tag{1.8}$$

Let π be an irreducible admissible representation of GSp(4, F). Following [J. PS. S] we say that π is a *Howe-lift* of σ if

$$\mathrm{Hom}_{\mathrm{GSp}(4,F)\times \mathrm{GSO}(6,F)}(\omega \otimes (\pi \otimes \hat{\sigma}), \mathbf{C}) \neq 0 \tag{1.9}$$

(ω is the appropriate local Weil representation; See section 0).

The functional l_ψ in (1.8) enters in the question of the uniqueness of a generic Howe lift of σ to GSp(4, F). Denote by $[\sigma]$ the set of equivalence classes of *generic* Howe lifts of σ to GSp(4, F).

THEOREM 1.3. *Let σ be an irreducible admissible representation of* GSO(6, F). *Then the cardinality of* $[\sigma]$ *is less than or equal to the dimension of the space of functionals* l_ψ.

PROOF. The proof is a local analogue of the proof of Proposition 1.1. Let π be an irreducible admissible representation of GSp(4, F) which is a generic Howe lift of σ. Then by (1.9) we get a morphism $\omega \otimes \hat{\sigma} \to \hat{\pi}$ which has the appropriate equivariance properties. Composing this morphism with the Whittaker functional of $\hat{\pi}$ we get a bilinear form of $S(Z_F^+ \times F^*) \times V_{\hat{\sigma}}$ ($V_{\hat{\sigma}}$ — the space of $\hat{\sigma}$) satisfying

$$(\omega(1, h)\phi, \hat{\sigma}(h)v) = (\phi, v); \qquad h \in \mathrm{GSO}(6, F), \quad v \in V_{\hat{\sigma}}, \tag{1.10}$$

$$\left(\omega\left[\begin{bmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{bmatrix}, 1\right]\phi, v\right) = \psi(x+y)(\phi, v) \tag{1.11}$$

Here $\phi \in S(Z_F^+ \times F^*)$ — the space of ω. We take Z^+ as in the proof of Proposition 1.1 so that we have the formulas (1.5) (locally). Put $E = S(Z_F^+ \times F^*)$. (1.11) means that (,) is a bilinear form on $E_{S,\psi} \times V_{\hat{\sigma}}$, where

$$S = \left\{\begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \in \mathrm{GSp}(4, F)\right\}, \qquad \psi\begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} = \psi\left(\mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} X\right)$$

and $E_{S,\psi}$ denotes the Jacquet module of E with respect to the group S and the character ψ. (See [B.Z], section 2.30.) Denote by $\omega_{S,\psi}$ the representation of the paraLolic subgroup

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

of GSp(4, F) in $E_{S,\psi}$ which is obtained from ω. Put

$$X_0 = \left\{ (x_1, x_2; t) \in Z_F^+ \times F^* \mid t \cdot \mathrm{Gr}(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

X_0 is closed in $Z_F^+ \times F^*$. Let P act in $S(X_0)$ according to the formulas of ω. We have an isomorphism of representations of P, $\omega_{S,\psi} \simeq S(X_0)$. It is given by $\alpha : \phi \to \mathrm{Res}_{X_0} \phi$. α is well defined. The exact sequence ([B.Z], section 1.8)

$$0 \to S(Z_F^+ \times F^* \backslash X_0) \to S(Z_F^+ \times F^*) \xrightarrow{\text{restriction}} S(X_0) \to 0$$

shows that α is an isomorphism. Thus we may think of (,) as a bilinear form of $S(X_0) \times V_{\hat{\sigma}}$ satisfying

$$(1.12) \qquad (\omega(1, h)\phi, \hat{\sigma}(h)v) = (\phi, v),$$

$$(1.13) \qquad \left(\omega \left[\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1 \end{bmatrix}, 1 \right] \phi, v \right) = \psi(u)(\phi, v),$$

with similar notations as in (1.10), (1.11), and we write ω for $\omega_{S,\psi}$. GSO(6, F) acts in $S(X_0)$ by left translations. Let A be the direct product

$$\left\{ \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1 \end{bmatrix} \,\middle|\, u \in F \right\} \times \mathrm{GSO}(6, F)$$

and consider the representation $\psi^{-1} \otimes \hat{\sigma}$ of A on $V_{\hat{\sigma}}$. The space of bilinear forms satisfying (1.12), (1.13) is isomorphic to $I(\sigma, \psi) = \mathrm{Hom}_A\,(\psi^{-1} \otimes \hat{\sigma}, S_A^*(X_0))$. ($S_A^*(X_0)$ denotes the space of A smooth distributions on X_0, i.e., distributions on X_0 which have open stabilizers in A.) As in the proof of Proposition 1.1, X_0 is the union of two orbits under A, $O_0 \cup O_1$. The elements of O_0 are of the form $(0, x; t)$. The elements $(x_1, x_2; t)$ of O_1 have the property that x_1, x_2 are linearly independent. Since O_0 is closed in X_0, we have by ([B.Z]) the exact sequence

$$0 \to \mathrm{Hom}_A(\psi^{-1} \otimes \hat{\sigma}, S_A^*(O_0)) \to I(\sigma; \psi) \to \mathrm{Hom}_A(\psi^{-1} \otimes \hat{\sigma}, S_A^*(O_1)).$$

For $\phi \in S(O_0)$, we have

$$\omega\left[\begin{bmatrix}1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1\end{bmatrix}, 1\right]\phi = \phi,$$

then $\mathrm{Hom}_A(\psi^{-1} \otimes \hat{\sigma}, S_A^*(O_0)) = 0$. Thus we have an injection

$$I(\sigma; \psi) \mapsto \mathrm{Hom}_A(\psi^{-1} \otimes \hat{\sigma}, S_A^*(O_1)).$$

Now take $0 \neq l \in \mathrm{Hom}_A(\psi^{-1} \otimes \hat{\sigma}, S_A^*(O_1))$. Let $v \in V_{\hat{\sigma}}$ and l_v its image under l in $S_A^*(O_1)$. Take the representative $(e_0, e; 1)$ for O_1 that we used in Proposition 1.1. Since its stabilizer and A are unimodular, then l_v is determined by an A-smooth function φ_v on O_1, once we fix an invariant measure dy on O_1. We have

$$l_v(\phi) = \int_{O_1} \varphi_v(y)\phi(y)dy. \tag{1.14}$$

This implies that

(1.15)
$$\psi(x)\varphi_v(y) = |\mu(h)|^{-3}\varphi_{\hat{\sigma}(h)v}\left(h \cdot y \cdot \begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right); \quad h \in \mathrm{GSO}(6, F), \quad y \in O_1.$$

This shows that φ_v is determined by the linear functional

$$P(v) = \varphi_v(e_0, e; 1)$$

which, by (1.15), satisfies the condition (1.8). The proof of the theorem is now complete. □

In the next section we show that the space of linear functions (1.8) is of dimension at most one. This will prove:

COROLLARY. *Let σ be an irreducible admissible representation of* GSO(6, F) *and assume it has a generic Howe-lift to* GSp(4, F), *then this lift is unique.*

Going back to the global case let us consider the injectivity property of the θ-correspondence from generic representations of GSp(4, A) to GSO(6, A).

THEOREM 1.4. *Let π_1, π_2 be two irreducible, automorphic cuspidal, generic representations of* GSp(4, A). *Let $\theta(\pi_i)$, $i = 1, 2$, denote the θ-lift of π_i to* GSO(6, A). *Assume that $\theta(\pi_i)$ are cuspidal and that $\theta(\pi_1) = \theta(\pi_2)$, then $\pi_1 = \pi_2$.*

PROOF. We should remark first that $\theta(\pi_i)$ is irreducible. Indeed if $\theta(\pi_i) = \oplus_\alpha \sigma_{i,\alpha}$ is a direct sum decomposition to irreducible, automorphic cuspidal representations of $GSO(6, A)$, then all the $\sigma_{i,\alpha}$ are locally equivalent at almost all places (see [H.PS]). By the strong multiplicity one theorem ([J.Sh]) all the $\sigma_{i,\alpha}$ are isomorphic and by the multiplicity one theorem ([Sh]) there is only one summand in the decomposition. (Recall that $PGSO(6) = PGL(4)$.) Put $\theta(\pi_1) = \theta(\pi_2) = \sigma$. Then

$$(1.16) \qquad \int_{GSO(6,k)\backslash GSO(6,A)} \int_{GSp(4,k)\backslash GSp(4,A)} \theta^{\phi}(g, h)\varphi(g)\overline{f(h)}dgdh \not\equiv 0$$

where $\varphi \in \pi_i$ and $f \in \sigma$. (1.16) implies that

$$\mathrm{Hom}_{GSp(4,A)\times GSO(6,A)}(\omega \otimes \pi_i \otimes \hat{\sigma}, \mathbf{C}) \neq 0, \qquad i = 1, 2.$$

This implies that there is a place v such that

$$\mathrm{Hom}_{GSp(4,k_v\times GSO(6,k_v)}(\omega_v \otimes \pi_{i,v} \otimes \hat{\sigma}_v, \mathbf{C}) \neq 0,$$

i.e., that $\pi_{i,v}$ is a generic Howe-lift of σ_v to $GSp(4, k_v)$. In [J.PS.S] it is shown that if v is archimedean then $\pi_{i,v}$ is uniquely determined by the parameters of σ_v. (See the following remark.) By the Corollary to Theorem 1.3 it follows that for v nonarchimedean $\pi_{1,v} \simeq \pi_{2,v}$. Thus π_1 and π_2 are isomorphic and hence equal by the multiplicity one theorem for generic representations of $GSp(4, A)$ ([PS]). □

REMARK. We sketch the proof of [J.PS.S] of the injectivity in the archimedean case. Put $k_v = F$, $\hat{\sigma}_v = \sigma$, $\pi_{i,v} = \pi$, $\omega_v = \omega$. Consider the action of ω on $C_c^\infty(F^*)\overline{\otimes} S(Z_F^+)$ ($\overline{\otimes}$-inductive tensor product). Then we are given a continuous trilinear form T on $(C_c^\infty(F^*)\overline{\otimes} S(Z_F^+))\overline{\otimes} V_\pi \hat{\otimes} V_\sigma$, satisfying

$$T(\omega(g, h)(\varphi \otimes \phi) \otimes \pi(g)v \otimes \sigma(h)u) = T((\varphi \otimes \phi) \otimes v \otimes u).$$

(V_π, V_σ are the respective subspaces of smooth vectors.) This with $h = 1$ and

$$g = \begin{pmatrix} I & \\ & \lambda I \end{pmatrix}$$

shows that there is a (nonzero) trilinear form T_1 on $S(Z_F^+)\overline{\otimes} V_\pi \overline{\otimes} V_\sigma$, such that

$$T((\varphi \otimes \phi) \otimes v \otimes u) = T_1\left(\int_{F^*} \varphi(x)\phi \otimes \pi\begin{pmatrix} I & \\ & xI \end{pmatrix} v \otimes u d^*x\right).$$

T_1 then satisfies $T_1(\omega_1(g, h)\phi \otimes \pi(g)v \otimes \sigma(h)u) = T_1(\phi \otimes v \otimes u)$, where we restrict g to be in Sp(4, F), and $\omega_1(1, h)$ acts by left translation on ϕ. Now, π is a quotient of a minimal principal series representation ρ of GSp(4, F), induced by a (quasi) character ξ of B, the Borel subgroup. Replace π by ρ^∞. Identify the functions in ρ^∞ with their restrictions to Sp(4, F). By Frobenius reciprocity (Theorem 5.3.2.1 in [W]), we get a bilinear form $\tilde{T}_1$ on $S(Z_F^+) \otimes V_\sigma$ such that

$$\tilde{T}_1(\omega_1(b, h)\phi \otimes \sigma(h)u) = \delta_B^{1/2}\xi^{-1}\left(b\begin{pmatrix} I & \\ & \mu(h)I \end{pmatrix}\right)\tilde{T}_1(\phi \otimes u),$$

for $b \in B \cap \mathrm{Sp}(4, F)$.

Considering the action of $\omega_1(b, 1)$ with

$$b = \begin{pmatrix} I & S \\ & I \end{pmatrix},$$

and realizing $Z_F^+ = X_F \oplus X_F$, it can be shown that when regarding $\tilde{T}_1$ as a distribution on Z_F^+ with values in $V_{\hat{\sigma}}$, then it is supported on $X_0 = \{(x_1, x_2) \mid \mathrm{Gr}(x_1, x_2) = 0\}$, with no transversal derivatives. X_0 is the union of four orbits under

$$\left\{ \begin{bmatrix} * & * & & \\ 0 & * & & \\ & & * & 0 \\ & & * & * \end{bmatrix} \in \mathrm{Sp}(4, F) \right\} \times \mathrm{GSO}(6, F).$$

One orbit is open. The restriction of $\tilde{T}_1$ to the open orbit maps σ (via Frobenius reciprocity) to $\mathrm{Ind}_{B'}^{\mathrm{GSO}(6,F)}\xi'$ where B' is the Borel subgroup and ξ' is a character determined by ξ (and vice versa). The restriction to one of the remaining small orbits maps σ to some $\rho = \mathrm{Ind}_P^{\mathrm{GSO}(6,F)}\tau$, where $P \supsetneq B'$ and τ is a finite dimensional representation of the parabolic subgroup P. This is impossible since ρ cannot contain a generic representation. (σ is a local component of a cuspidal representation of GSO(6, A) and so σ is generic.) Since ξ is determined by (ξ' and hence by) σ, then π is determined by σ.

As an application we get

THEOREM 1.5 (The strong multiplicity one theorem for generic representations of GSp(4, A)). *Let π_1, π_2 be two irreducible, automorphic, cuspidal, generic representations of* GSp(4, A). *Write $\pi_i = \otimes_v \pi_{i,v}$ and assume that $\pi_{1,v} \cong \pi_{2,v}$ for almost all v, then $\pi_1 = \pi_2$.*

PROOF. Let $\theta(\pi_i)$ denote that θ-lift of π_i to GSO(6, A). Since the constituents of $\theta(\pi_i)$ are all locally equivalent almost everywhere then either $\theta(\pi_1)$, $\theta(\pi_2)$ are both cuspidal or both noncuspidal (since $\mathrm{PGSO}(6)=\mathrm{PGL}(4)$ and $Z_2\backslash \mathrm{GL}(4)\hookrightarrow \mathrm{GSO}(6)$, see section 0). Assume first that $\theta(\pi_i)$ are cuspidal and hence by the strong multiplicity one theorem (for GL(4)) $\theta(\pi_1)=\theta(\pi_2)$. By Theorem 1.4 $\pi_1=\pi_2$. Assume now that $\theta(\pi_i)$ are noncuspidal. Then (Rallis theorem [R1]) the θ-lift of π_i to $\mathrm{GSO}(4,A)$ is cuspidal, where GSO(4) is the connected component of the group of similitudes of a split symmetric form in four variables. (Recall that $\mathrm{GSO}(4)=\mathrm{GL}(2)\times \mathrm{GL}(2)/C$ where C denotes the scalars embedded diagonally.) Denote again by $\theta(\pi_i)$ the θ-lift of π_i to $\mathrm{GSO}(4,A)$. By the same reasoning as for the previous case $\theta(\pi_1)=\theta(\pi_2)$. Denote $\theta(\pi_i)=\sigma$. Then as in the proof of Theorem 1.4, there is a place v such that

$$\mathrm{Hom}_{\mathrm{GSp}(4,k_v)\times \mathrm{GSO}(4,k_v)}(\omega_v\otimes\pi_{i,v}\otimes\hat{\sigma}_v,\mathbf{C})\neq 0.$$

As in [J.PS.S], when v is archimedean, $\pi_{i,v}$ is completely determined by σ_v (see last remark). For v nonarchimedean an analogous proof to that of Theorem 1.3 shows that the number of generic Howe lifts of σ_v to $\mathrm{GSp}(4,k_v)$ is less than or equal to the dimension of the space of Whittaker functionals of σ_v which equals one. (see Theorem 3.1 in [S].) This shows that $\pi_{1,v}\cong\pi_{2,v}$ for all v and hence $\pi_1=\pi_2$. □

§2. The uniqueness theorem for the functional l_ψ

In this section F denotes a local nonarchimedean field, and $G=\mathrm{GSO}(6,F)$. We formulate our main theorem.

THEOREM 2.1. *Let σ be an irreducible, admissible representation of G, then the space of linear functionals* (1.8), *l_ψ, for σ, is of dimension at most one.*

In our proof we follow the Gelfand–Kazhdan method (see [B.Z]). Let us sketch it and give the details later. We first introduce an involution $g\to g^\tau$ on G which has the properties

$$R^\tau=R, \tag{2.1}$$

$$\chi_0(r^\tau)=\chi_0(r),\qquad \forall r\in R \tag{2.2}$$

(χ_0 is defined in (1.3)).

Next, we prove the following theorem, where F can be any field and ψ any nontrivial character of F.

THEOREM 2.2. *One of the following conditions holds for $g\in G$.*

(a) *There are* $r_1, r_2 \in R$ *such that*

$$r_1 g r_2^{-1} = g \quad \textit{and} \quad \psi(\chi_0(r_1 r_2^{-1})) \neq 1.$$

(b) *There are* $r_1, r_2 \in R$ *such that*

$$r_1 g r_2^{-1} = g^{\tau} \quad \textit{and} \quad \chi_0(r_1 r_2^{-1}) = 0.$$

The proof of Theorem 2.2 involves technical work. Now let σ be an irreducible admissible representation of G and let l_1, l_2 be two linear functionals on the space of σ, with the property (1.8). Define for $\varphi \in S(G)$ (the Schwartz functions on G)

$$B(\varphi) = l_2(\varphi * l_1') \tag{2.3}$$

where

$$l_1'(v) = l_1\left[\sigma\begin{bmatrix} -1 & & \\ & I & \\ & & -1 \end{bmatrix} v\right], \qquad v \in V_\sigma.$$

(l_1' has the property (1.8) with respect to ψ^{-1}.) For a linear functional l on V_σ, $\varphi * l$ is the vector in V_σ obtained as follows. Consider the linear functional

$$T_{\varphi,l}(v) = \int_G \varphi(g) l(\sigma(g)v)\,dg.$$

$T_{\varphi,l}$ is smooth and hence belongs to the space of the contragradient representation $\hat{\sigma}$ of σ (realized in the space of smooth linear functionals on V_σ). But $\hat{\sigma} \cong \omega_\sigma^{-1} \circ \mu \otimes \sigma$ where ω_σ is the central character of σ and $g \to \mu(g)$ is the similitude factor of g. Fix, then, a bilinear form $\langle \ , \ \rangle$ on $V_\sigma \times V_\sigma$ (it is unique up to a scalar), which has the property

$$\langle \sigma(g)v, \omega_\sigma^{-1}(\mu(g))\sigma(g)w \rangle = \langle v, w \rangle \qquad \text{for } v, w \in V_\sigma, \quad g \in G.$$

We define $\varphi * l \in V_\sigma$ by the relation $T_{\varphi,l}(v) = \langle \varphi * l, v \rangle$. Note that if ρ, λ denote respectively the right and left translation representations of G in $S(G)$, then

$$(\rho(g)\varphi) * l = \omega_\sigma^{-1}(\mu(g))\sigma(g)(\varphi * l), \tag{2.4}$$

$$(\lambda(g)\varphi) * l = \varphi * \check{\sigma}(g^{-1}) l \tag{2.5}$$

($\check{\sigma}$ denotes the algebraic dual of σ). In particular, we have

$$B(\rho(r)\varphi) = \psi(\chi_0(r))B(\varphi), \tag{2.6}$$

$$B(\lambda(r)\varphi) = \psi^{-1}(\chi_0(r))B(\varphi), \qquad r \in R. \tag{2.7}$$

We will use Theorem 2.2 and the Gelfand–Kazhdan theorem (Theorem 6.10 in [B.Z]) to conclude:

THEOREM 2.3. *The distribution B is τ-invariant. (The action of τ on B is by $B^\tau(\varphi) = B(\varphi^\tau)$, and $\varphi^\tau(g) = \varphi(g^\tau)$.)*

We now specify the definition of τ. Recall that G acts from the left on the space X and write the elements of G as matrices with respect to the decomposition (1.1). Let α be the reflection on X with respect to e, that is, $\alpha \cdot e = -e$ and $\alpha \cdot v = v$ for all $v \in X$ orthogonal to e (e enters in the definition of R and χ_0 in (1.2), (1.3)). We define for $g \in G$

$$g^\tau = \mu(g)\alpha^{-1}g^{-1}\alpha. \tag{2.8}$$

Clearly $(g^\tau)^\tau = g$ and $(g_1g_2)^\tau = g_2^\tau g_1^\tau$. To check (2.1), we note that R is characterized by the fact that its elements preserve e_0 and send e to a vector of the form $te_0 + e$. Since α preserves e_0 and sends e to $-e$, it is clear that (2.1) is satisfied. For (2.2), let $r \in R$ satisfy $r \cdot e = e + te_0$. By (2.8),

$$r^\tau \cdot e = -\alpha r^{-1} e = -\alpha \cdot (e - te_0) = e + te_0.$$

Thus $r \cdot e = r^\tau \cdot e$ and hence $\chi_0(r) = \chi_0(r^\tau) = -t$. Theorem 2.1 now follows in a standard way.

PROOF OF THEOREM 2.1. We first need a lemma.

LEMMA 2.4. *The τ-invariance of the distribution B implies that*

$$\{\varphi \in S(G) \mid \varphi * l_1' = 0\} = \{\varphi \in S(G) \mid ((\omega_\sigma^{-1} \circ \mu) \otimes \varphi^{\alpha,\mu}) * l_2 = 0\}$$

where $((\omega_\sigma^{-1} \circ \mu) \otimes \varphi^{\alpha,\mu})(g) = \omega_\sigma^{-1}(\mu(g))\varphi(\mu^{-1}(g)\alpha g \alpha)$.

Applying the lemma to the distribution $B'(\varphi) = l_2(\varphi * l_2')$ we get that

$$\{\varphi \in S(G) \mid \varphi * l_2' = 0\} = \{\varphi \in S(G) \mid ((\omega_\sigma^{-1} \circ \mu) \otimes \varphi^{\alpha,\mu}) * l_2 = 0\}.$$

Put $J_i = \{\varphi \in S(G) \mid \varphi * l_i' = 0\}$, $i = 1, 2$. Then $J_1 = J_2$. Let ρ denote the right translations of G in $S(G)$. The map $A_i : \varphi \to \varphi * l_i'$ defines an isomorphism of representations

$$(\rho, S(G)/J_i) \xrightarrow{\sim} (\omega_\sigma^{-1} \circ \mu \otimes \sigma, V_\sigma).$$

It is injective by definition, it intertwines the representations by (2.4), and it is surjective by the irreducibility of σ. Now define $T: V_\sigma \to V_\sigma$ by $T(\varphi * l_1') = \varphi * l_2'$. T is well defined since $J_1 = J_2$, and it is an automorphism of $\omega_\sigma^{-1} \circ \mu \otimes \sigma$.

This implies that $T = \delta \cdot \mathrm{id}$ for $\delta \in \mathbf{C}$. Thus $\varphi * l_2' = \delta \cdot \varphi * l_1'$ for all $\varphi \in S(G)$. This implies that $l_2' = \delta \cdot l_1'$ and hence $l_2 = \delta \cdot l_1$.

PROOF OF LEMMA 2.4. First note that for a linear functional l on V_σ and $\varphi \in S(G)$, we have

$$\varphi * l = \int_G \varphi(g)\omega_\sigma(\mu(g))\sigma(g^{-1})\tilde{l}^{K_\varphi}dg \tag{2.9}$$

where K_φ is a small compact open subgroup of G, depending on φ, and $\tilde{l}^{K_\varphi}$ is defined by $l^{K_\varphi}(v) = \langle \tilde{l}^{K_\varphi}, v \rangle$, $\forall v \in V_\sigma$, where

$$l^{K_\varphi}(v) = \frac{1}{m(K_\varphi)} \int_{K_\varphi} l(\sigma(k)v)dk.$$

$m(K_\varphi)$ is the measure of K_φ. Indeed, let K_φ be a small compact open subgroup of G satisfying $\varphi(kg) = \varphi(g)$, $\forall g \in G$, $k \in K_\varphi$. Then for $v \in V_\sigma$, we have

$$\begin{aligned}
\langle \varphi * l, v \rangle &= \int_G \varphi(g) l(\sigma(g)v)dg \\
&= \frac{1}{m(K_\varphi)} \int_{K_\varphi} \int_G \varphi(kg) l(\sigma(k)\sigma(g)v)dgdk \\
&= \int_G \varphi(g) l^{K_\varphi}(\sigma(g)v)dg \\
&= \int_G \varphi(g)\omega_\sigma(\mu(g))\langle \sigma(g^{-1})\tilde{l}^{K_\varphi}, v \rangle dg.
\end{aligned}$$

This implies (2.9). To prove the lemma, assume that $\varphi * l_1' = 0$. By (2.4) $\rho(g)\varphi * l_1' = 0$ for all $g \in G$. By the τ-invariance of B, we get that

$$B((\rho(g)\varphi)^\tau) = 0$$

for all $g \in G$. Take $g^0 \in G$ and write it in the form $g^0 = \mu^{-1}(g_0)g_0$. We have

$$\begin{aligned}
0 &= l_2((\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau * l_1') \\
&= \int_G (\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau(g)\omega_\sigma(\mu(g)) l_2(\sigma(g^{-1})\tilde{l}_1'^{K_{\varphi,g_0}})dg \\
&= \int_G (\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau(g)\omega_\sigma(\mu(g))\langle \tilde{l}_2^{K_\varphi'}, \sigma(g^{-1})\tilde{l}_1'^{K_{\varphi,g_0}}\rangle dg
\end{aligned}$$

($K_{\varphi,g_0} = K_{(\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau}$, and K_φ' is a small compact open subgroup of G satisfying $\varphi(\mu(k)\alpha k^{-1}\alpha g) = \varphi(g)$ for all $k \in K_\varphi'$ and $g \in G$. We also assume that $\omega_\sigma(\mu(k)) = 1$ for $k \in K_\varphi'$).

$$= \pm \int_G (\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau(g)\omega_\sigma(\mu(g))\langle\sigma(g^{-1})\tilde{l}_1'^{K_\varphi, g_0}, \tilde{l}_2^{K'_\varphi}\rangle dg.$$

(It is clear that $\langle v_1, v_2\rangle = \delta\langle v_2, v_1\rangle$ for all $v_1, v_2 \in V_\sigma$ and that $\delta = \pm 1$.) Thus

$$0 = \int_G (\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau(g)\omega_\sigma(\mu(g))\langle\sigma(g^{-1})\tilde{l}_1'^{K_\varphi, g_0}, \tilde{l}_2^{K'_\varphi}\rangle dg$$

$$= \int_G (\rho(\mu^{-1}(g_0)g_0)\varphi)^\tau(g) l_1'(\sigma(g)\tilde{l}_2^{K'_\varphi})dg$$

$$= \int_G \varphi^\alpha(\mu(gg_0^{-1})g^{-1}\alpha g_0\alpha) l_1'(\sigma(g)\tilde{l}_2^{K'_\varphi})dg$$

$$= \int_G \varphi^\alpha(\mu(g)g^{-1}) l_1'(\sigma(g_0^\alpha g)\tilde{l}_2^{K'_\varphi})dg$$

$$= l_1'(\sigma(g_0^\alpha)\int_G \varphi^\alpha(\mu(g)g^{-1})\sigma(g)\tilde{l}_2^{K'_\varphi}dg).$$

Since this is true for any $g_0 \in G$, then

$$0 = \int_G \varphi^\alpha(\mu(g)g^{-1})\sigma(g)\tilde{l}_2^{K'_\varphi}dg$$

$$= \int_G \varphi^\alpha(\mu^{-1}(g)g)\sigma(g^{-1})\tilde{l}_2^{K'_\varphi}dg$$

$$= \int_G \omega_\sigma^{-1}(\mu(g))\varphi^{\alpha,\mu}(g)\omega_\sigma(\mu(g))\sigma(g^{-1})\tilde{l}_2^{K'_\varphi}dg$$

$$= (\omega_\sigma^{-1} \circ \mu \otimes \varphi^{\alpha,\mu}) * l_2.$$

By reversing the steps we get the desired equality. This proves the lemma and Theorem 2.1, using Theorem 2.3. □

PROOF OF THEOREM 2.3. We verify the assumptions of Theorem (6.10) in [B.Z]. Put $H = R \times R$. Let H act on G by $(r_1, r_2)\cdot g = r_1 g r_2^{-1}$, and on $S(G)$ by $(r_1, r_2)\cdot\varphi(g) = \psi^{-1}(\chi_0(r_1^{-1}r_2)\varphi(r_1^{-1}gr_2)$.

The assumptions of Theorem (6.10) of [B.Z] in this case are the following:

(a) The action of H on G is constructive (i.e., the set $\{(g, h\cdot g) \mid g \in G, h \in H\}$ is the union of finitely many locally closed subsets of $G \times G$).

(b) For each $h \in H$, there is $h_\tau \in H$ such that $h \cdot g^\tau = (h_\tau \cdot g)^\tau$ for all $g \in G$.

(c) $\tau^2 = \mathrm{id}$.

(d) If T is a nonzero H-invariant distribution on an H-orbit Y, then $Y^\tau = Y$ and $T^\tau = T$.

The conclusion is that any H-invariant distribution on G is also τ-invariant. Note that by (2.6), (2.7) our distribution B is H-invariant.

The condition (a) is implied by Theorem A in 6.15 of [B.Z]. For (b), we have

$$(r_1, r_2)\cdot g^\tau = \mu(g)r_1\alpha g^{-1}\alpha r_2^{-1} = \mu(g)\alpha((\alpha r_2\alpha)g(\alpha r_1^{-1}\alpha))^{-1}\alpha = ((\alpha r_2\alpha, \alpha r_1\alpha)\cdot g)^\tau.$$

(2.1) implies that for $(r_1, r_2)\in H$, $(\alpha r_2\alpha, \alpha r_1\alpha)$ is also in H. Assumption (c) is immediate. The verification of (d) requires some work and is linked with Theorem 2.2. Let T be a nonzero H-invariant distribution on an H-orbit $Y = H\cdot g$. This means that

$$T(\lambda(r_1)\rho(r_2)\varphi) = \psi(\chi_0(r_1^{-1}r_2))T(\varphi) \qquad \text{for } \varphi\in S(Y).$$

Let H_g denote the stabilizer of g in H. Denote the character of H, $(r_1, r_2)\to \psi(\chi_0(r_1^{-1}r_2))$ by $\tilde{\psi}$, then since $S(Y)\cong \mathrm{Ind}^{cH}_{H_g}1$ (compact induction), we have that

$$T\in\mathrm{Hom}_H(\mathrm{Ind}^{cH}_{H_g}1, \tilde{\psi})\cong \mathrm{Hom}_{H_g}(\Delta_{H_g}/\Delta_H, \mathrm{Res}_{H_g}\tilde{\psi})$$

by the Frobenius reciprocity, where Δ_{H_g}, Δ_H are the modular functions of H_g and H. Note that $\Delta_H = 1$.

Lemma 2.5. $\Delta_{H_g} = 1$ *for all* $g\in G$ (*i.e.*, H_g *is unimodular*).

We will prove the lemma later. By the lemma, we have to consider the space $\mathrm{Hom}_{H_g}(1, \mathrm{Res}_{H_g}\tilde{\psi})$. Thus, if $\mathrm{Res}_{H_g}\tilde{\psi}\neq 1$ then $T=0$. Since T is nonzero, we must have $\mathrm{Res}_{H_g}\tilde{\psi} = 1$. By Theorem 2.2, only the possibility (b) there is valid for g, which means that the orbit $Y = H\cdot g$ is τ-invariant. This proves one part of (d). It remains to show that $T^\tau = T$. In our case T is proportional (see 6.12 of [B.Z]) to

$$T_g(\varphi) = \int_{H_g\backslash H}\varphi(h^{-1}\cdot g)\tilde{\psi}^{-1}(h)dh$$

where dh is a right H-invariant meassure on $H_g\backslash H$. Let $h_0\in H$ satisfy $h_0^{-1}\cdot g = g^\tau$ and $\tilde{\psi}(h_0) = 1$ (Theorem 2.2 (b)). We have

$$T_g^\tau(\varphi) = T_g(\varphi^\tau) = \int_{H_g\backslash H}\varphi((h^{-1}\cdot g)^\tau)\tilde{\psi}^{-1}(h)dh.$$

Write $(h^{-1}\cdot g)^\tau = (\tilde{h}^\tau)^{-1}\cdot g^\tau$ where if $h = (r_1, r_2)$ then $\tilde{h}^\tau = (r_2^{-\tau}, r_1^{-\tau})$. So we get

$$T_g^\tau(\varphi) = \int_{H_g\backslash H}\varphi((h_0\tilde{h}^\tau)^{-1}\cdot g)\tilde{\psi}^{-1}(h)dh.$$

Put $h_0 = (r_1^0, r_2^0)$ and $h_0^* = (r_2^{0\tau}, r_1^{0\tau})$. Note that $\tilde{h}_0^{*\tau} = h_0^{-1}$ and $(h_0^*)^{-1}\cdot g = g^\tau$. The change of variables $h\to\beta(h) = h_0^*\tilde{h}^\tau$ is permissible here. Indeed if $h_1\in H_g$ then

$$\beta(h_1h) = h_0^*(\tilde{h}_1h^\tau) = h_0^*\tilde{h}_1^\tau\tilde{h}^\tau = h_0^*\tilde{h}_1^\tau(h_0^*)^{-1}h_0^*\tilde{h}^\tau \in H_g h_0^*\tilde{h}^\tau = H_g\beta(h).$$

Using (2.2), we see that $\tilde{\psi}(\beta(h)) = \psi(h)$. Thus, using that H_g is unimodular, we get

$$\begin{aligned} T_g^\tau(\varphi) &= \int_{H_g\backslash H} \varphi((h_0\beta(\tilde{h})^\tau)^{-1}\cdot g)\tilde{\psi}^{-1}(h)dh \\ &= \int_{H_g\backslash H} \varphi(h^{-1}\cdot g)\tilde{\psi}^{-1}(h)dh \\ &= T_g(\varphi). \end{aligned}$$

This proves part (d).

Proof of Lemma 2.5. We compute H_g for $g \in G$.

It is enough to do it for representatives of $R\backslash G/R$. Choose a basis e_1, e_2, e_3, e_4 for L (in the notation of (1.1)) such that the matrix $((e_i, e_j))$, $1 \leqq i, j \leqq 4$ is equal to

$$w = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

We write the elements of G according to the basis $e_0, e_1, \ldots, e_4, e_{-0}$. We find three types of representative for $R\backslash G/R$.

(1) $$g = \begin{bmatrix} x & & \\ & b & \\ & & y \end{bmatrix}, \qquad b \in \mathrm{GSO}(L) = \mathrm{GSO}(4, F) \quad (\mu(b) = xy),$$

(2) $$g = \begin{bmatrix} x & & \\ & b & \\ & & y \end{bmatrix} w_2 \begin{bmatrix} x' & & \\ & b' & \\ & & y' \end{bmatrix}; \quad w_2 = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix},$$

(zeros elsewhere)

(3) $$g = \begin{bmatrix} x & & \\ & b & \\ & & y \end{bmatrix} w_3 \begin{bmatrix} x' & & \\ & b' & \\ & & y' \end{bmatrix}; \quad w_3 = \begin{bmatrix} & & 1 \\ & \alpha & \\ 1 & & \end{bmatrix},$$

where α denotes the restriction of the reflection α to L. Take g of type (1). Assume that $(r_1, r_2) \in H_g$. Write

$$r_i = \begin{bmatrix} 1 & -{}^t\boldsymbol{v}_i \cdot wh_i & -\frac{1}{2}(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ & h_i & \boldsymbol{v}_i \\ & & 1 \end{bmatrix}.$$

Then $r_1 g = g r_2$ is equivalent to

$$h_1 = b h_2 b^{-1}, \tag{2.10}$$

$$\boldsymbol{v}_1 = y^{-1} b \cdot \boldsymbol{v}_2. \tag{2.11}$$

Thus $\boldsymbol{v}_1$ is determined by $\boldsymbol{v}_2$ and (2.10) means that h_2 belongs to the subgroup D of $\mathrm{SO}(L)$ which preserves the vectors $\boldsymbol{e}$ and $b^{-1} \cdot \boldsymbol{e}$. Thus $H_g \cong D \cdot L$ which is unimodular.

Now let g be of type (2) and let $(r_1, r_2) \in H_g$. Write r_i as before and

$$w_2 = \begin{bmatrix} 0 & {}^t\boldsymbol{e}_1 & 0 \\ \boldsymbol{e}_1 & M & \boldsymbol{e}_2 \\ 0 & {}^t\boldsymbol{e}_4 & 0 \end{bmatrix}$$

according to the decomposition (1.1) where

$$M = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}.$$

Then $r_1 g = g r_2$ implies that

$$\begin{bmatrix} 1 & -{}^t\boldsymbol{v}_1 \cdot w & -\frac{1}{2}(\boldsymbol{v}_1, \boldsymbol{v}_1) \\ & I_4 & \boldsymbol{v}_1 \\ & & 1 \end{bmatrix} \begin{bmatrix} x & & \\ & h_1 b & \\ & & y \end{bmatrix} w_2 \begin{bmatrix} x' & & \\ & b' h_2^{-1} & \\ & & y' \end{bmatrix}$$

$$= g \begin{bmatrix} 1 & -{}^t\boldsymbol{v}_2 \cdot w & -\frac{1}{2}(\boldsymbol{v}_2, \boldsymbol{v}_2) \\ & I_4 & \boldsymbol{v}_2 \\ & & 1 \end{bmatrix}. \tag{2.12}$$

Equating both sides of (2.12) we get the following equations:

$${}^t\boldsymbol{v}_1 w h_1 b \boldsymbol{e}_1 = 0, \tag{2.13}$$

$$h_1 b \boldsymbol{e}_1 = b \boldsymbol{e}_1, \tag{2.14}$$

$${}^t\boldsymbol{e}_4 b' h_2^{-1} = {}^t\boldsymbol{e}_4 b', \tag{2.15}$$

$$(2.16)\qquad {}^te_4 b' v_2 = 0,$$

$$(2.17)\qquad -y'\,{}^tv_1 w h_1 b e_4 = x\,{}^te_1 b' v_2,$$

$$(2.18)\qquad x\,{}^te_1 b' h_2^{-1} - {}^tv_1 w h_1 b M b' h_2^{-1} - \tfrac{1}{2}(v_1, v_1) y\,{}^te_4 b' h_2^{-1} = x\,{}^te_1 b',$$

$$(2.19)\qquad h_1 b M b' h_2^{-1} + y v_1\,{}^te_4 b' h_2^{-1} = -x' b e_1\,{}^tv_2 w + bMb',$$

$$(2.20)\qquad y' h_1 b e_4 = -\tfrac{1}{2}(v_2, v_2) x' b e_1 + bMb' v_2 + y' b e_4.$$

We will show that the solution of this system of equations is the following: First, the expressions of v_1 and h_1 via g, v_2 and h_2 are

$$(2.18)'\qquad -{}^tv_1 w b = x\,{}^te_1 + (y'^{-1} x\,{}^te_1 b' h_2^{-1} v_2 + x\,{}^te_1 b' h_2^{-1} b'^{-1} e_4)\,{}^te_4 - x\,{}^te_1 b' h_2^{-1} b'^{-1},$$

$$(2.20)'\qquad b^{-1} h_1 b = \begin{bmatrix} 1 & -x'(v_2, h_2 b'^{-1} e_2) & -x'(v_2, h_2 b'^{-1} e_3) & -\tfrac{1}{2} y'^{-1} x'(v_2, v_2) \\ 0 & z & 0 & y'^{-1}(e_3, b' v_2) \\ 0 & 0 & z^{-1} & y'^{-1}(e_2, b' v_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{\text{def}}{=} T(r_2);$$

h_2 and v_2 satisfy the following conditions:

$$(2.15)'\qquad b' h_2^{-1} b'^{-1} = \begin{bmatrix} 1 & * & * & * \\ 0 & z^{-1} & 0 & * \\ 0 & 0 & z & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(and of course, } h_2 e = e\text{)},$$

$$(2.16)\qquad {}^te_4 b' v_2 = 0,$$

$$(2.21)\qquad T(r_2) b^{-1} e = b^{-1} e.$$

We note that (2.15)′ is another form of (2.15), and (2.21) follows from (2.20)′. Now (2.15)′ says that h_2 belongs to the subgroup of $\mathrm{SO}(L)$ which fixes e and the isotropic vector $w\,{}^tb' e_4$. Equations (2.16), (2.21) give conditions on v_2. Indeed, let

$$b^{-1} e = \begin{bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_4 \end{bmatrix},$$

then by (2.21) and (2.20)′

$$\begin{cases} -(\nu_2, h_2 b'^{-1} e_2)\delta_2 - (\nu_2, h_2 b'^{-1} e_3)\delta_3 - \frac{1}{2} y'^{-1}(\nu_2, \nu_2)\delta_4 = 0, \\ z\delta_2 + y'^{-1}(e_3, b'\nu_2)\delta_4 = \delta_2, \\ z^{-1}\delta_3 + y'^{-1}(e_2, b'\nu_2)\delta_4 = \delta_3. \end{cases}$$

If $\delta_4 = 0$, then $\delta_2\delta_3 \neq 0$, $((e, e) = 1)$, and hence $(z = 1$ and)

$$(\nu, h_2 b'^{-1} e_3) = -\delta_3^{-1}\delta_2(\nu_2, h_2 b'^{-1}\nu_2).$$

Thus ν_2 is of the form

$$\nu_2 = h_2 b'^{-1}\begin{bmatrix} c_1 \\ c_2 \\ -\delta_2^{-1}\delta_3 c_2 \\ 0 \end{bmatrix}; \qquad c_1, c_2 \in F.$$

If $\delta_4 \neq 0$ then $(e_3, b'\nu_2) = y'\delta_4^{-1}\delta_2(1 - z)$ and $(e_3, b'\nu_2) = y'\delta_4^{-1}\delta_3(1 - z^{-1})$ and so

$$\nu_2 = b'^{-1}\begin{bmatrix} c \\ y'(1 - z)\delta_4^{-1}\delta_2 \\ y'(1 - z^{-1})\delta_4^{-1}\delta_3 \\ 0 \end{bmatrix}, \qquad z = {}^t e_2 b' h_2 b'^{-1} e_2, \quad c \in F.$$

Now it is easy to see that in the first case $dc_1 dc_2 dh_2$ and in the second $dc dh_2$ are unimodular measures on H_g.

It remains to show (2.18)′ and (2.20)′.

PROOF OF (2.18)′. We have

$$-{}^t\nu_1 w h_1 b M b' h_2^{-1} \overset{(2.19)}{=} -{}^t\nu_1 w(-y\nu_1 {}^t e_4 b' h_2^{-1} - x' b e_1 {}^t\nu_2 w + bMb')$$
$$\overset{(2.14)+(2.13)}{=} y(\nu_1, \nu_1){}^t e_4 b' h_2^{-1} - {}^t\nu_1 w b M b'.$$

Substitute in (2.18):

$$(*) \qquad x' {}^t e_1 b' h_2^{-1} + \tfrac{1}{2} y(\nu_1, \nu_1){}^t e_4 b' h_2^{-1} - {}^t\nu_1 w b M b' = x' {}^t e_1 b'.$$

We also have

$$-y' {}^t\nu_1 w h_1 b e_4 \overset{(2.20)}{=} -{}^t\nu_1 w(-\tfrac{1}{2}(\nu_2, \nu_2) x' b e_1 + bMb'\nu_2 + y' b e_4)$$
$$\overset{(2.13)+(2.14)}{=} -{}^t\nu_1 w b M b' \nu_2 - y' {}^t\nu_1 w b e_4.$$

Substitute in (2.17):

$$(**) \qquad -{}^t v_1 wbMb'v_2 - y'{}^t v_1 wbe_4 = x{}^t e_1 b'v_2.$$

Multiply (*) by v_2 from the right, compare with (**), and use (2.15) and (2.16) to obtain

$$-{}^t v_1 wbe_4 = y'^{-1}x{}^t e_1 b'h_2^{-1}v_2.$$

Using this, (*), (2.14) + (2.13) (i.e., ${}^t v_1 wbe_1 = 0$) and (2.15)′ we get

$$(***) \qquad \begin{aligned} -{}^t v_1 wb &= (-{}^t v_1 wbe_1){}^t e_1 + (-{}^t v_1 wbM) + (-{}^t v_1 wbe_4){}^t e_4 \\ &= x{}^t e_1 + (y'^{-1}x{}^t e_1 b'h_2^{-1}v_2 - \tfrac{1}{2}y(v_1, v_1)){}^t e_4 - x{}^t e_1 b'h_2^{-1}b'^{-1}. \end{aligned}$$

Use this to compute (easily), using (2.15)′,

$$({}^t bwv_1, {}^t bwv_1) = -2x^2\,{}^t e_1 b'h_2^{-1}b'^{-1}e_4.$$

But $({}^t bwv_1, {}^t bwv_1) = xy(v_1, v_1)$ and so

$$-\tfrac{1}{2}y(v_1, v_1) = x{}^t e_1 b'h_2^{-1}b'^{-1}e_4.$$

Substitute this in (***) to get (2.18)′.

PROOF OF (2.20)′. (2.20) and (2.14) can be written together as follows:

$$h_1 b \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} = b \begin{bmatrix} 1 & 0 & 0 & -\tfrac{1}{2}y'^{-1}x'(v_2, v_2) \\ 0 & 0 & 0 & y'^{-1}(e_3, b'v_2) \\ 0 & 0 & 0 & y'^{-1}(e_2, b'v_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying (2.19) from the right and using (2.15)′, we get

$$h_1 bM = b \begin{bmatrix} 0 & -x'(v_2, h_2 b'^{-1}e_2) & -x'(v_2, h_2 b'^{-1}e_3) & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Adding the last two equalities gives (2.20)′.

Let g be of type (3). We can write g in the form

$$g = \begin{bmatrix} x & & \\ & b & \\ & & y \end{bmatrix} w_3 \qquad (\mu(b) = xy).$$

Let $(r_1, r_2) \in H_g$ and write

$$r_i = \begin{bmatrix} 1 & -\boldsymbol{v}_i{}^t w h_i & -\tfrac{1}{2}(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ & h_i & \boldsymbol{v}_i \\ & & 1 \end{bmatrix},$$

then $r_1 g = g r_2$ implies that $\boldsymbol{v}_1 = \boldsymbol{v}_2 = 0$ and $h_1 = b h_2 b^{-1}$ (note that $h_2\alpha = \alpha h_2$). Thus

$$H_g \cong \left\{ h \in \mathrm{SO}(L) \;\middle|\; \begin{matrix} he = e \\ h(b^{-1}e) = b^{-1}e \end{matrix} \right\}$$

which is unimodular. This proves Lemma 2.5 (and Theorem 2.3, using Theorem 2.2). □

Proof of Theorem 2.2. We use the same notation as for the proof of Lemma 2.5. It is enough to prove the theorem for representatives of $R\backslash G/R$. Let g be of type (1), and let $(r_1, r_2) \in H_g$ where

$$r_i = \begin{bmatrix} 1 & -{}^t\boldsymbol{v}_i w h_i & -\tfrac{1}{2}(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ & h_i & \boldsymbol{v}_i \\ & & 1 \end{bmatrix} \in R$$

then (2.10, 2.11), $h_1 = b h_2 b'^{-1}$ and $\boldsymbol{v}_1 = y^{-1} b \boldsymbol{v}_2$. Thus

$$\begin{aligned} \chi_0(r_1 r_2^{-1}) &= (\boldsymbol{v}_1, e) - (\boldsymbol{v}_2, e) = (y^{-1} b \boldsymbol{v}_2, e) - (\boldsymbol{v}_2, e) = x(\boldsymbol{v}_2, b^{-1}e) - (\boldsymbol{v}_2, e) \\ &= (\boldsymbol{v}_2, x b^{-1} e - e). \end{aligned}$$

If $b^{-1}e \neq x^{-1}e$ then there is $\boldsymbol{v}_2 \in L$ such that $\psi((\boldsymbol{v}_2, x b^{-1}e - e)) \neq 1$ and we are in case (a) of the theorem. If $b^{-1}e = x^{-1}e$ then, since $\mu(b) = xy$ we get $x = y$ and so

$$g = x \begin{bmatrix} 1 & & \\ & x^{-1}b & \\ & & 1 \end{bmatrix}, \qquad (x^{-1}b)e = e.$$

Thus $g \in x \cdot R$, hence we may assume that $b = xI_4$ and $g = xI_6$. We have $g^\tau = \mu(g)\alpha g^{-1}\alpha = x^2 x^{-1} I_6 = xI_6 = g$, and so we are in case (b) of the theorem. Let now g be of type (2). Assume that $(u_1, u_2) \in H_g$ where

$$u_i = \begin{bmatrix} 1 & -{}^t\boldsymbol{v}_i w & \tfrac{1}{2}(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ & I_4 & \boldsymbol{v}_i \\ & & 1 \end{bmatrix}.$$

Then by (2.18)′, we have

$$-{}^t\boldsymbol{v}_1 w b = (y'^{-1} x\, {}^t e_1 b' \boldsymbol{v}_2)^t e_4 = y'^{-1} x (e_4, b' \boldsymbol{v}_2)^t e_4$$

and hence

$$v_1 = -(yy')^{-1}(e_4, b'v_2)be_1.$$

By (2.16), (2.20)′, we have

$$v_2 = cb'^{-1}e_1, \qquad c \in F.$$

Thus $v_1 = -(yy')^{-1}cbe_1$, $v_2 = cb'^{-1}e_1$, and so

$$\chi_0(u_1u_2^{-1}) = (v_1, e) - (v_2, e) = -c((y')^{-1}be_1 + b'^{-1}e_1, e).$$

Thus if $((yy')^{-1}be_1 + b'^{-1}e_1, e) \neq 0$, we can find $c \in F$ such that $\psi(\chi_0(u_1u_2^{-1})) \neq 1$ and we are in case (a). Now, assume that

$$((yy')^{-1}be_1 + b'^{-1}e_1, e) = 0. \tag{2.22}$$

We show that g satisfies condition (b) of the theorem. So let us solve the equations

$$\begin{cases} r_1gr_2^{-1} = g^\tau, \\ \chi_0(r_1r_2^{-1}) = 0. \end{cases} \tag{2.23}$$

Write

$$r_i = \begin{bmatrix} 1 & -{}^tv_iwh_i & -\frac{1}{2}(v_i, v_i) \\ & h_i & v_i \\ & & 1 \end{bmatrix} \in R.$$

We look at the first equation of (2.23). Write it in the form

$$\begin{aligned} &\begin{bmatrix} 1 & -{}^tv_1wh_1 & \frac{1}{2}(v_1, v_1) \\ & h_1 & v_1 \\ & & 1 \end{bmatrix} \begin{bmatrix} x & & \\ & b & \\ & & y \end{bmatrix} w_2 \begin{bmatrix} x' & & \\ & b'h_2^{-1} & \\ & & y' \end{bmatrix} \\ &= \mu(g)\alpha g^{-1}\alpha \begin{bmatrix} 1 & -{}^tv_2w & \frac{1}{2}(v_2, v_2) \\ & I_4 & v_2 \\ & & 1 \end{bmatrix}. \end{aligned} \tag{2.24}$$

Equating both sides of (2.24), we get the following equations:

$$x'h_1be_1 = \mu(bb')x^{-1}\alpha b'^{-1}e_1, \tag{2.25}$$

$$y\,{}^te_4b'h_2^{-1} = \mu(bb')y'^{-1}\,{}^te_4b^{-1}\alpha, \tag{2.26}$$

$${}^tv_1wh_1be_1 = 0, \tag{2.27}$$

$$ {}^{t}e_4 b^{-1}\alpha v_2 = 0, \tag{2.28}$$

$$ -y'\,{}^{t}v_1 w h_1 b e_4 = \mu(bb')x'^{-1}\,{}^{t}e_1 b^{-1}\alpha v_2, \tag{2.29}$$

$$ \begin{aligned} & x\,{}^{t}e_1 b' h_2^{-1} - {}^{t}v_1 w h_1 b M b' h_2^{-1} - \tfrac{1}{2}y(v_1 v_1)\,{}^{t}e_4 b' h_2^{-1} \\ & = \mu(bb')x'^{-1}\,{}^{t}e_1 b^{-1}\alpha, \end{aligned} \tag{2.30}$$

$$ \begin{aligned} & h_1 b M b' h_2^{-1} + y v_1\,{}^{t}e_4 b' h_2^{-1} \\ & = -x^{-1}\mu(bb')\alpha b'^{-1} e_1\,{}^{t}v_2 w + \mu(bb')\alpha b'^{-1} M b^{-1}\alpha, \end{aligned} \tag{2.31}$$

$$ \begin{aligned} & \mu^{-1}(bb')y' h_1 b e_4 \\ & = -\tfrac{1}{2}x^{-1}(v_2, v_2)\alpha b'^{-1} e_1 + \alpha b'^{-1} M b^{-1}\alpha v_2 + y^{-1}\alpha b'^{-1} e_4. \end{aligned} \tag{2.32}$$

We can find h_1 satisfying (2.25) if and only if $x'(be_1, e) = \mu(bb')x^{-1}(\alpha b'^{-1}e_1, e)$. Since $\mu(bb') = xyx'y'$ and $\alpha e = -e$ we get $((yy')^{-1}be_1 + b'^{-1}e_1, e) = 0$, which is condition (2.22). Equation (2.26) is the same as (2.25) for h_2^{-1}. Indeed, write (2.26) as follows:

$$ y\,{}^{t}e_1 w b' h_2^{-1} = \mu(bb')y'^{-1}\,{}^{t}e_1 w b^{-1}\alpha, $$

then

$$ y\,{}^{t}h_2^{-1}\,{}^{t}b' w e_1 = \mu(bb')y'^{-1}\,{}^{t}\alpha\,{}^{t}b^{-1} w e_1. $$

Shift w to the left and cancel it, then

$$ y\mu(b')h_2 b'^{-1} e_1 = \mu(bb')\mu^{-1}(b)y'^{-1}\alpha b e_1. $$

Since α commutes with h_2, we get

$$ h_2^{-1} b e_1 = yy'\alpha b'^{-1} e_1 $$

which is the equation (2.25) for h_2^{-1}. So we choose $h_2^{-1} = h_1$.

Write $v_1 = \Sigma_{i=1}^4 t_i h_1 b e_i$ and $v_2 = \Sigma_{i=1}^4 z_i \alpha b e_1$. We show that the solution of the system of equations is $v_2 = -\alpha h_1^{-1} v_1$, and h_1, v_1 should satisfy (2.25), (2.31). Note that (2.27) says that $t_4 = 0$, (2.28) says that $z_4 = 0$ and (2.29) says that $z_1 = -t_1$. Consider now equation (2.30).

We have

$$ \begin{aligned} \mu(bb')x'^{-1}\,{}^{t}e_1 b^{-1}\alpha h_2 b'^{-1} &= x\,{}^{t}e_1 - {}^{t}v_1 w h_1 b M - \tfrac{1}{2}y(v_1, v_1)\,{}^{t}e_4 \\ &= x\,{}^{t}e_1 - \sum_{i=1}^{3} t_i\,{}^{t}e_i\,{}^{t}b\,{}^{t}h_1 w h_1 b M - y\mu(b)t_2 t_3\,{}^{t}e_4 \\ &= x\,{}^{t}e_1 - \mu(b)t_3\,{}^{t}e_2 - \mu(b)t_2\,{}^{t}e_3 - y\mu(b)t_2 t_3\,{}^{t}e_4 \end{aligned} $$

hence

$$(2.30)' \qquad {}^t e_1 b^{-1}\alpha h_2 b'^{-1} = (yy')^{-1}\,{}^t e_1 - y'^{-1} t_3\,{}^t e_2 - y'^{-1} t_2\,{}^t e_3 - yy'^{-1} t_2 t_3\,{}^t e_4.$$

Consider equation (2.32):

$$\mu^{-1}(bb')y'b'\alpha h_1 b e_4 = \tfrac{1}{2}x^{-1}(v_2, v_2)e_1 + Mb^{-1}\alpha v_2 + y^{-1}e_4$$
$$= -yz_2z_3e_1 + \sum_{i=1}^{3} z_i M e_i + y'^{-1}e_4 = -yz_2z_3e_1 + z_2e_2 + z_3e_3 + y^{-1}e_4.$$

We have

$${}^t(\mu^{-1}(bb')y'b'\alpha h_1 b e_4) = \mu^{-1}(bb')y'\,{}^t e_1 w\,{}^t(b'\alpha h_1 b) = y'\,{}^t e_1 b^{-1} h_1^{-1}\alpha b'\alpha w$$
$$= y'\,{}^t e_1 b^{-1}\alpha h_2 b'^{-1} w,$$

since $h_2 = h_1^{-1}$ and α commutes with h_1. Thus we get

$$(2.32)' \qquad {}^t e_1 b^{-1}\alpha h_2 b'^{-1} = -yy'^{-1} z_2 z_3\,{}^t e_4 + y'^{-1} z_3\,{}^t e_2 + y'^{-1} z_2\,{}^t e_3 + (yy')^{-1}\,{}^t e_1.$$

Comparing with (2.30)′ we get $z_2 = -t_2$ and $z_3 = -t_3$. Thus $v_2 = -\alpha h_1^{-1} v_1$, where $v_1 = \Sigma_{i=1}^{3} t_i h_1 b e_i$ and h_1 satisfies (2.25).

It remains to satisfy equation (2.31). Write it as follows:

$$(2.31)' \qquad \begin{aligned} &y v_1\,{}^t e_4 b' h_2^{-1} + x^{-1}\mu(bb')\alpha b'^{-1} e_1\,{}^t v_2 w \\ &= \mu(bb')\alpha b'^{-1} M b^{-1}\alpha - h_1 b M b' h_2^{-1}. \end{aligned}$$

The left side of (2.31) equals, using (2.25), (2.26),

$$\mu(bb')y'^{-1} v_1\,{}^t e_4 b^{-1}\alpha - x' h_1 b e_1\,{}^t v_1\,{}^t h_1^{-1}\,{}^t\alpha w$$
$$= \mu(bb')y'^{-1} h_1 b \sum_{i=1}^{3} t_i e_i\,{}^t e_4 b^{-1}\alpha - x' h_1 b e_1 \sum_{i=1}^{3} t_i\,{}^t e_i\,{}^t b\,{}^t\alpha w$$
$$= \mu(b)x' h_1 b \left(\sum_{i=1}^{3} t_i (e_i\,{}^t e_4 - e_1\,{}^t e_i w)\right) b^{-1}\alpha$$
$$= \mu(b)x' h_1 b \begin{bmatrix} 0 & -t_3 & -t_2 & 0 \\ 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & t_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} b^{-1}\alpha.$$

Thus (2.31)′ can we written as follows (recall that $h_2 = h_1^{-1}$):

$$(2.31)'' \qquad \mu(b)x' \begin{bmatrix} 0 & -t_3 & -t_2 & 0 \\ 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & t_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mu(bb')(b'h_1\alpha b)^{-1}M - M(b'h_1\alpha b).$$

By (2.26)

$$(b'h_1\alpha b)^{-1} = \left[\begin{array}{c|cc|c} (yy')^{-1} & -d_3 & -d_2 & f \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{A}} & c_2 \\ 0 & & & c_3 \\ \hline 0 & 0 & 0 & (xx')^{-1} \end{array}\right]$$

where $A \in \mathrm{GO}(2, F)$ (the group of similitudes of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). Since $\det \alpha = -1$ and $b, b' \in \mathrm{GSO}(L)$ (i.e., $\det b = \mu(b)^2$), we must have $\det A = -\mu(A)$ and so A must be of the form

$$A = \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix}.$$

We have

$${}^t e_1(b'h_1\alpha b)^{-1} = (yy')^{-1}\,{}^t e_1 - d_3\,{}^t e_2 - d_2\,{}^t e_3 + f\,{}^t e_4$$

and by (2.30)′ $(d_2, d_3) = y'^{-1}(t_2, t_3)$ and hence

$$A^{-1}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = yy'\begin{pmatrix} d_2 \\ d_3 \end{pmatrix} = y\begin{pmatrix} t_2 \\ t_3 \end{pmatrix}.$$

Thus the right hand side of (2.31)″ equals

$$\mu(bb')\left[\begin{array}{c|cc|c} 0 & -d_3 & -d_2 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ 0 & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{A^{-1}}} & \multirow{2}{*}{$-xx'A^{-1}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix}$} \\ 0 & & & \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|cc|c} 0 & -\mu(b)x't_3 & -\mu(b)x't_2 & 0 \\ \hline 0 & & & \mu(b)x't_2 \\ & \mu(A)^{-1}A & -A^{-1} & \\ 0 & & & \mu(b)x't_3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

We have

$$\mu(A)^{-1} - A^{-1} = (rs)^{-1}\begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} - \begin{pmatrix} 0 & s^{-1} \\ r^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This shows that equation (2.31)″ is satisfied. We have

$$\chi_0(r_1r_2^{-1}) = (v_1, e) - (v_2, e) = (-\alpha h_1^{-1}v_2, e) - (v_2, e)$$
$$= (h_1^{-1}v_2, e) - (v_2, e) = (v_2, e) - (v_2, e) = 0.$$

This proves (2.33).

Let g be of type (3). We show that it satisfies condition (b) of the theorem. Write g in the form

$$g = \begin{bmatrix} s & & \\ & b & \\ & & y \end{bmatrix} w_3.$$

We find

$$r_i = \begin{bmatrix} 1 & & \\ & h_i & \\ & & 1 \end{bmatrix} \in R$$

such that

$$(2.33) \qquad r_1gr_2^{-1} = g^{\tau}.$$

Note that $\chi_0(r_1r_2^{-1}) = 0$. Equation (2.33) is equivalent to

$$(2.33)' \qquad \begin{cases} h_1bh_2^{-1} = \mu(b)b^{-1}, \\ h_ie = e, \qquad i = 1, 2. \end{cases}$$

It is easy to see that there is $h_1 \in \mathrm{SO}(L)$ which satisfies

$$\begin{cases} h_1e = e \\ h_1be = \mu(b)b^{-1}e. \end{cases}$$

Thus h_1 and $h_2 = \mu^{-1}(b)bh_1b$ solve (2.33)′. This completes the proof of Theorem 2.2. □

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